ON THE IRREDUCIBLE REPRESENTATION ALGEBRA OF THE ALTERNATING GROUP OF DEGREE FOUR

V. BOVDI, V. RUDKO

Abstract. We obtain a description of the irreducible representation algebra of the alternating group of degree four over the ring of 2-adic integers.

Let G be a finite group and let K be a commutative principal ideal domain. We denote the module M of a K-representation of the group G by the symbol $[M]$. We assume that $[M] = [N]$ if and only if the KG-modules M and N of K-representations are isomorphic. The ring $\mathfrak{a}(KG)$ of K-representation of the group G is a smallest ring which contains all symbols $[M]$ with the following operations

$$[M] + [N] = [M \oplus N], \qquad [M] \cdot [N] = [M \otimes N],$$

where $\quad g(m \otimes n) = g(m) \otimes g(n), \quad (g \in G,\, m \in M,\, n \in N)$.

The subring $\mathfrak{b}(KG) \subset \mathfrak{a}(KG)$ which is generated by $[M]$, where M is a module of the irreducible K-representation of the group G is called the irreducible K-representations ring of G.

The algebras $\mathfrak{A}(KG) = \mathfrak{a}(KG) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathfrak{B}(KG) = \mathfrak{b}(KG) \otimes_{\mathbb{Z}} \mathbb{Q}$ over the field of rational numbers $\mathbb{Q}$ are called the K-representation algebra of G and the irreducible K-representation algebra of G, respectively.

It is well known that $\mathfrak{A}(RG)$, where R is the ring of p-adic integers, is finite dimensional if and only if the p-Sylow subgroups of G are cyclic of order p^r, where $r \leq 2$. This is a classical result of S.D. Berman, P.M. Gudivok, A. Heller and I. Reiner (see Theorem 33.6, [2], p.690).

Although the number of irreducible R-representations of the group G is finite, the algebra $\mathfrak{B}(RG)$ may be infinite dimensional (see [1, 3, 4]).

In the present paper we obtain a description of the irreducible R-representation algebra of the alternating group A_4 of degree four over the ring R of 2-adic integers. Note that earlier it was known that this algebra is infinite dimensional (see Proposition 13.1, [4], p.108).

Our main result is the following.

Theorem. *Let R be the ring of 2-adic integers. The irreducible R-representation algebra $\mathfrak{B}(RG)$ of the alternating group G of degree 4 isomorphic to*

$$\mathbb{Q}[x, x^{-1}] \oplus \mathbb{Q}[x, x^{-1}] \oplus \mathbb{Q}[x, x^{-1}] \oplus \mathbb{Q}[x, x^{-1}].$$

Notation. Let G be the alternating group of degree four. Put $a_1 = (1,2)(3,4)$, $a_2 = (1,4)(2,3)$, $\quad b = (1,2,3) \in S_4$. Clearly

$$G \cong \langle a_1, b\rangle = H \rtimes \langle b\rangle \cong (C_2 \times C_2) \rtimes C_3,$$

1991 *Mathematics Subject Classification.* Primary: 20C11; Secondary: 16A64, 20C10, 20G05.
Key words and phrases. integral representation, algebra representation.
The research was supported by OTKA No.K68383.

where $H=\langle a_1,a_2\rangle\cong C_2\times C_2$ is the 2-Sylow subgroup of G.

Let L be a free RG-module over the ring R with R-basis $\{e_1,e_2,e_3,e_4\}$ in which G acts as a permutation group. Let d be a divisor of 4 and put

$$\begin{aligned} L_d &= \Big\{\sum_{j=1}^{4}\alpha_j e_j \;\mid\; \alpha_j\in R,\quad \sum_{j=1}^{4}\alpha_j\equiv 0 \pmod d\Big\},\\ L_0 &= R(e_1+\cdots+e_4),\qquad M_d=L_d/L_0.\end{aligned}$$

Obviously L_d, L_0 and M_d are RG-modules and the elements

$$u_1=de_1+L_0,\qquad u_2=e_2-e_1+L_0,\qquad u_3=e_3-e_2+L_0$$

form an R-basis of M_d. Since $e_4\equiv -e_1-e_2-e_3 \pmod{L_0}$, it is not difficult to show that the following R-representation of G

$$\Gamma_d: a_1\mapsto \begin{pmatrix}1&0&-4d^{-1}\\ d&-1&-2\\ 0&0&-1\end{pmatrix},\qquad a_2\mapsto\begin{pmatrix}-3&4d{-}1&0\\ -2d&3&0\\ -d&2&-1\end{pmatrix},\qquad b\mapsto\begin{pmatrix}1&0&0\\ d&0&-1\\ 0&1&-1\end{pmatrix}$$

is afforded by module the M_d.

Note that Γ_2 is equivalent to the following monomial representation

$$a_1\mapsto\begin{pmatrix}-1&0&0\\ 0&1&0\\ 0&0&-1\end{pmatrix},\qquad b\mapsto\begin{pmatrix}0&0&1\\ 1&0&0\\ 0&1&0\end{pmatrix}.$$

We shall need the following result.

Lemma 1. *(see [5]) The representations Γ_d, where $d=1,2,4$, are irreducible and nonequivalent R-representations of the group G. Moreover, except these representations and the trivial one $\tau_0: g\mapsto 1$ ($g\in G$), the group G has only one more irreducible R-representation:*

$$\tau: a_1\mapsto E,\qquad b\mapsto\begin{pmatrix}0&-1\\ 1&-1\end{pmatrix}.$$

Lemma 2. *Let $\Gamma=P_0\oplus P_1$, where $P_0=RGw_0$, $P_1=RGw_1$ are projective RG-modules and $w_0=\frac{1}{3}(1+b+b^2)$, $w_1=1-w_0$ are orthogonal idempotents. Then the following equations hold*

$$[P_0]^2=2[P_0]+[P_1],\qquad [P_0][P_1]=2[P_0]+3[P_1],\qquad [P_1]^2=6[P_0]+5[P_1].$$

Proof. It is easy to see that $\chi_{P_0}(b)=1$ and $\chi_{P_1}(b)=-1$, where χ_{P_i} is the character of P_i. Remember that $\chi_{P_0\oplus P_1}=\chi_{P_0}+\chi_{P_1}$, $\quad\chi_{P_0\otimes P_1}=\chi_{P_0}\chi_{P_1}$.

If $[P_0]^2=s[P_0]+t[P_1]$, then $s-t=1$ and $4s+8t=16$. It follows that $s=2$ and $t=1$, so the first equation is true. The proof of second one is analogues. $\square$

Corollary 1. *The elements*

$$\mathfrak{f}_1=\tfrac{1}{12}[\Gamma],\qquad \mathfrak{f}_2=[P_0](1-\mathfrak{f}_1),\qquad \mathfrak{f}_3=(1-[P_0])(1-\mathfrak{f}_1)\tag{1}$$

are pairwise orthogonal idempotents of the algebra $\mathfrak{A}(RG)$ such that $\mathfrak{f}_1+\mathfrak{f}_2+\mathfrak{f}_3=1$. Moreover,

$$\mathfrak{A}(KG)\mathfrak{f}_1\cong\mathfrak{A}(KG)\mathfrak{f}_2\cong\mathbb{Q}.$$

Let W be a finite group. Assume that for the K-representations of W the Krull-Schmidt-Azumaya theorem holds (see Theorem 36.0, [2], p.768). We assume that there exists at least one nonzero indecomposable, non-projective KW-module of the K-representation of the group W. For each module B of the K-representation

of the group W we consider the K-module $B^* = Hom_K(B, K)$. Clearly B^* is a KW-module of the K-representation of the group W such that

$$(g \cdot \varphi) \cdot b = \varphi(g^{-1}b), \qquad (g \in W,\ \varphi \in B^*,\ b \in B).$$

The module B^* is called the contragredient module of B.

Note that each module of the K-representation of W is always a homomorphic image of a free module, and it can be considered as a submodule of a free module, too.

We shall use the following well known result.

Lemma 3. *Let $0 \to B \to P \to A \to 0$ be an exact sequence of modules of K-representations of the group W. If P is a projective KW-module and A is an indecomposable non-projective KW-module, then the module B contains only one unique indecomposable nonprojective direct summand B_0. Moreover the sequence $0 \to B_0 \to P_0 \to A \to 0$ is exact for some projective KW-module P_0. There exists a duality corresponding to the exchange of A and B.*

Proof. All modules in the lemma are free of finite rank over the ring K. There exists a projective KW-module P and a sequence of indecomposable KW-modules $B_1, \dots, B_s$, such that the sequence $0 \to B_1 \oplus \cdots \oplus B_s \to P \to A \to 0$ is exact.

Assume that $A_1, \dots, A_s$ are KW-modules and $P_1, \dots, P_s$ are projective KW-modules, such that the sequences $0 \to B_j \to P_j \to A_j \to 0$ are exact for each $1 \le j \le s$. Sum them over j and apply Schanuel's Lemma (Lemma 2.24, [2], p.30), then we obtain that

$$P \oplus A_1 \oplus \cdots \oplus A_s \cong A \oplus P_1 \oplus \cdots \oplus P_s.$$

This yields that A is a direct summand only one of the A_i, says of A_1, so $A_2, \dots, A_s$ are projective. Then the modules $B_2, \dots, B_s$ are projective, too. Therefore we have the following exact sequence $0 \to B_0 \oplus P' \to P \to A \to 0$, where P' is projective and B_0 is an indecomposable non-projective KW-module. The following isomorphism of KW-modules hold

$$A \cong P/(B_0 \oplus P') \cong (P/P')/B_0,$$

where P/P' is a projective KW-module.

By the use of the contragredient module we obtain the dual statement. □

Obviously, each projective RH-module is free, where $H = Syl_2(G)$. Let M be a module of R-representation of the group H and assume that M does not contain projective summands. By $\Theta(M)$ we denote the kernel (the Green operator, after J.A. Green) of the projective cover of RH-module M (see Lemma 3). The following sequence of RH-modules

$$0 \to \Theta(M) \to F \to M \to 0$$

is exact, where F is a free RH-module of smallest rank. If M is an indecomposable RH-module, then $\Theta(M)$ is indecomposable, too.

Let Δ_0 be the module of the trivial representation $h \mapsto 1 \quad (h \in H)$ of H. Put

$$\Delta_n = \Theta(\Delta_{n-1}), \qquad \Delta_{-n} = \Delta_n^*, \qquad (n = 1, 2, \dots)$$

where Δ_n^* is the contragredient module to Δ_n.

Lemma 4. *The sequence* $0 \to \Delta_n \to (RH)^n \to \Delta_{n-1} \to 0$ *is exact and*

$$\operatorname{rank}_R(\Delta_n) = 2n+1.$$

Proof. Let Γ^n be a n-dimensional module over the ring RH, where $H = \langle a_1, a_2\rangle$. In the follows we will treat Γ^n considering as component columns. Denote by F_n the matrix over the ring RH with n rows and $n+1$ columns, in which only the three upper diagonals (if we begin calculation from the main diagonal) are non-zero:

- the main diagonal of F_n is the first n element from the following sequence $\digamma_1 \cup \digamma_1 \cup \digamma_1, \dots$, where
$$\digamma_1 = \begin{cases} \{a_1-1, a_2-1, -(a_1+1), -(a_2+1)\} & \text{if } n \text{ is odd},\\ \{a_1+1, a_2+1, -(a_1-1), -(a_2-1)\} & \text{if } n \text{ is even}.\end{cases}$$
- the second diagonal consist of zero elements except the last one which is equal to one of the following elements
$$a_2-1,\ -(a_1-1),\ a_2+1,\ a_1+1$$
according to the cases $n \equiv \{1,2,3,0\} \pmod 4$, respectively.
- the third diagonal consist of $n-1$ elements of the sequence $\digamma_2 \cup \digamma_2 \cup \digamma_2, \dots$, where $\digamma_2 = \{a_2-1, a_1-1, a_2+1, a_1+1\}$.

Example.
$$F_4 = \begin{pmatrix} a_1+1 & 0 & a_2-1 & 0 & 0\\ 0 & a_2+1 & 0 & a_1-1 & 0\\ 0 & 0 & -(a_1-1) & 0 & a_2+1\\ 0 & 0 & 0 & -(a_2-1) & a_1+1\end{pmatrix};$$
$$F_5 = \begin{pmatrix} a_1-1 & 0 & a_2-1 & 0 & 0 & 0\\ 0 & a_2-1 & 0 & a_1-1 & 0 & 0\\ 0 & 0 & -(a_1+1) & 0 & a_2+1 & 0\\ 0 & 0 & 0 & -(a_2+1) & 0 & a_1+1\\ 0 & 0 & 0 & 0 & a_1-1 & a_2-1\end{pmatrix}.$$

Let V_n be a submodule in Γ^n generated by the columns of the matrix F_n. Let $\varepsilon : \Gamma^n \to V_{n-1}$ be the following epimorphism of modules:

$$\Gamma^n \ni x = (x_1, \dots, x_n) \mapsto x_1 f_1 + \dots + x_n f_n \in V_n,$$

where f_j are columns of the matrix F_{n-1}.

It is not difficult to check that $F_{n-1}F_n = 0$.

Moreover, any solution of $\varepsilon(x) = 0$ belongs to the linear combination of the columns of F_n, so $x \in V_n$. Therefore, we proved the exactness of the sequence

$$0 \to V_n \to \Gamma^n \to V_{n-1} \to 0.$$

We construct a basis of the R-module V in the following way: the first $n+1$ elements of the basis are the columns of F_n. Then we take the first n columns of F_n, which multiply each of them by corresponding element of the sequence $\digamma_3 \cup \digamma_3 \cup \ldots \cup \digamma_3$, where $\digamma_3 = \{a_2-1, a_1-1, a_2+1, a_1+1\}$. These columns form the remaining n elements of the basis.

Consequently we obtain a system of $2n+1$ basis elements of R-module V_n. By Lemma 3, beginning with V_0 all RH-modules V_n are indecomposable. □

Induce the exact sequence of Lemma 4 from the subgroup H to the group G. In this exact sequence of RG-modules the middle modules are free, and the 2^{nd} and the 4^{th} terms of the sequence have a decomposition into the direct sums of indecomposable RG-modules

$$\Delta_n^G = \Delta_{0,n} \oplus \Delta_{1,n},$$

where $\operatorname{rank}(\Delta_{0,n}) = \operatorname{rank}(\Delta_n)$ and $\operatorname{rank}(\Delta_{1,n}) = 2 \cdot \operatorname{rank}(\Delta_n)$. Moreover $\Delta_{0,0}$ and $\Delta_{1,0}$ are modules of the R-representations τ_0, τ (see Lemma 1) of G, respectively. It is easy to check that $\Delta_{0,1}$ and $\Delta_{0,-1}$ are modules of the irreducible R-representations of Γ_1, Γ_4 of G, respectively (see Lemma 1). Since

$$(\Delta_n^G)|_H = (\Delta_n)^{(3)},$$

the RG-module $\Delta_{0,n}$ is a lifting of the RH-module Δ_n.

We will use the notation $\Delta_n = \Delta_{0,n}$.

Lemma 5. *The sequences of RG-modules*

$$0 \to \Delta_{3k} \to \Gamma^k \to \Delta_{3k-1} \to 0,$$
$$0 \to \Delta_{3k+1} \to P_0 \oplus \Gamma^k \to \Delta_{3k} \to 0,$$
$$0 \to \Delta_{3k+2} \to P_1 \oplus \Gamma^k \to \Delta_{3k+1} \to 0$$

are exact, where $\Gamma = RG = P_0 \oplus P_1$ (see Lemma 2). Moreover

$$\Delta_{3k} \otimes \Delta_1 = \Delta_{3k+1} \oplus \Gamma^k,$$
$$\Delta_{3k+1} \otimes \Delta_1 = \Delta_{3k+2} \oplus P_0 \oplus \Gamma^k,$$
$$\Delta_{3k+2} \otimes \Delta_1 = \Delta_{3k+3} \oplus P_1 \oplus \Gamma^k.$$

Proof. It is easy to see that

$$\{(a_1 - 1)(a_2 - 1)w_0,\ (a_1 - 1)w_0,\ (a_2 - 1)w_0\}$$

is an R-basis of the RG-submodule $M \subset P_0$. Obviously, $P_0/M \cong \Delta_0$, so we can assume that $M = \Delta_1$ and the following sequence

$$0 \to \Delta_1 \to P_0 \to \Delta_0 \to 0 \tag{2}$$

is exact. Comparing the values of the characters at $b \in G$, we get

$$P_0 \otimes \Delta_1 \cong \Gamma = P_0 \oplus P_1.$$

Multiplying (2) tensorially by Δ_1, we obtain the exact sequence

$$0 \to \Delta_1 \otimes \Delta_1 \to P_0 \oplus P_1 \to \Delta_1 \to 0,$$

which is possible only if $\Delta_1 \otimes \Delta_1 \cong \Delta_2 \oplus P_0$, so the sequence

$$0 \to \Delta_2 \to P_1 \to \Delta_1 \to 0.$$

is exact, too. Multiplying the last sequence again tensorially by Δ_1 and using the value of the characters (for example, $\chi_{\Delta_2}(b) = -1$), we can verify the lemma for $n = 2, 3, \ldots$. □

Using the contragredient modules we can obtain an analogue of Lemma 5 for the modules Δ_m with negative m. As a consequence we have

Corollary 2. *In the algebra $\mathfrak{A}(RG)$ the following equations hold*

$$[\Delta_n] \cdot [\Delta_m]\mathfrak{f}_3 = [\Delta_{n+m}]\mathfrak{f}_3, \qquad [\Delta_{1,0}] \cdot [\Delta_{1,0}] = 2[\Delta_0] + [\Delta_{1,0}].$$

Proof. These equations follows from Lemma 5, where $[P_j]\mathfrak{f}_3 = 0$ and $\mathfrak{f}_3$ from (1). □

The group $H = \langle a_1, a_2\rangle \cong C_2 \times C_2$ has the following four linear characters:

$$\begin{array}{llll} \delta_0: & a_1 \mapsto 1, \quad a_2 \mapsto \quad 1; & \delta_1: a_1 \mapsto -1, & a_2 \mapsto \ \ 1; \\ \delta_2: & a_1 \mapsto 1, \quad a_2 \mapsto \ -1; & \delta_3: a_1 \mapsto -1, & a_2 \mapsto -1. \end{array}$$

It is easy to check that the induced representations δ_1^G, δ_2^G, δ_3^G of G are irreducible and equivalent to the representation Γ_2 (see Lemma 1).

Lemma 6. *The following equations hold*

$$[\Delta_{1,0}]^2 = [\Delta_{1,0}] + [\Delta_0], \qquad [L]^2 = [\Delta_0] + [\Delta_{1,0}] + 2[L],$$

where L is the module of the representation Γ_2 (see Lemma 1) and Δ_0 is the trivial RG-module.

Proof. Using Mackey's theorem (see Theorem 10.18, [2], p.240) we get

$$\delta_2^G \otimes \delta_2^G \simeq \sum_{j=0}^{2} (\delta_2 \otimes \delta_2^{b^j})^G,$$

where $\delta_2^{b^j}(h) = \delta_2(b^{-j}hb^j)$ for $h \in H$. This yields that $\delta_2 \otimes \delta_2 = \delta_0$, $\delta_2 \otimes \delta_2^b = \delta_1$, $\delta_2 \otimes \delta_2^{b^2} = \delta_3$, and $\delta_2^G \otimes \delta_2^G = \delta_0^G + \delta_1^G + \delta_3^G$. $\square$

Proof of the Theorem. Let $\mathfrak{B}'(RG) = \mathfrak{B}(RG)\mathfrak{f}_3$. Put

$$x = [\Delta_1], \qquad y = [\Delta_{1,0}], \qquad z = [L].$$

We identify $\mathfrak{f}_3$ with the unity 1 of the algebra $\mathfrak{B}'(RG)$, where $\mathfrak{f}_3$ from (1). Since $[\Delta_1][\Delta_{-1}]\mathfrak{f}_3 = \mathfrak{f}_3$, we can assume that $[\Delta_{-1}]\mathfrak{f}_3 = \frac{1}{x}$. By Lemma 6 it follows that

$$\mathfrak{B}'(RG) = \langle\ 1,\ x,\ \tfrac{1}{x},\ y,\ z\ \mid\ y^2 = y + 2, \quad z^2 = 2z + y + 1\ \rangle.$$

Since $y^2 - y - 2 = (y-2)(y+1)$, we have

$$\langle\ 1,\ x,\ \tfrac{1}{x},\ y\ \rangle \cong \mathbb{Q}[x, \tfrac{1}{x}][y]/\langle y^2 - y - 2\rangle \cong \mathbb{Q}[x, \tfrac{1}{x}] \oplus \mathbb{Q}[x, \tfrac{1}{x}].$$

Now the algebra $\mathfrak{B}'(RG) \cong \mathbb{Q}[x, \frac{1}{x}] \oplus \mathbb{Q}[x, \frac{1}{x}] \oplus \mathbb{Q}[x, \frac{1}{x}] \oplus \mathbb{Q}[x, \frac{1}{x}]$, because

$$z^2 - 2z - y - 1 = \begin{cases} (z+1)(z-3) & \text{for} \quad y = 2; \\ (z-2)z & \text{for} \quad y = -1. \end{cases}$$

Finally, since the algebra $\mathfrak{B}(RG)$ has no projective summands, the map $u \mapsto u\mathfrak{f}_3$ gives the isomorphism $\mathfrak{B}'(RG) \cong \mathfrak{B}(RG)$. $\square$

References

[1] V. F. Barannik, P. M. Gudivok, and V. P. Rudko. On the tensor products of irreducible integral p-adic representations of finite p-groups. *Nauk. Vīsn. Uzhgorod. Univ. Ser. Mat.*, (1):5–13, 1994.

[2] C. Curtis and I. Reiner. *Methods of representation theory. Vol. I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1990. With applications to finite groups and orders, Reprint of the 1981 original, A Wiley-Interscience Publication.

[3] P. M. Gudivok, S. F. Gončarova, and V. P. Rudko. Tensor products of irreducible integral p-adic representations of finite groups. *Ukrain. Mat. Zh.*, 34(6):688–694, 813, 1982.

[4] P. M. Gudivok and V. P. Rudko. *Tensor products of representations of finite groups.* (Tenzornye proizvedeniya predstavlenij konechnykh grupp). Uzhgorod State University, Uzhgorod, 1985.

[5] L. A. Nazarova. Unimodular representations of the alternating group of degree four. *Ukrain. Mat. Ž.*, 15:437–444, 1963.

Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O.B. 12, Hungary

E-mail address: vbovdi@math.klte.hu